

Wave scattering by small impedance particles in a medium

A.G. Ramm

Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

The theory of acoustic wave scattering by many small bodies is developed for bodies with impedance boundary condition. It is shown that if one embeds many small particles in a bounded domain, filled with a known material, then one can create a new material with the properties very different from the properties of the original material. Moreover, these very different properties occur although the total volume of the embedded small particles is negligible compared with the volume of the original material.

1 Introduction

In [10] a theory of wave scattering by many small acoustically soft particles, embedded in a bounded domain D , filled with a material with known refraction coefficient, a medium, is developed. "Acoustically soft" means that the Dirichlet condition holds on the boundary of the small particles. Using the general methodology, developed in [10], we study here the wave scattering on impedance particles, derive a linear algebraic system for quantities which yield the scattered field if the number M of the embedded particles is of order 10, and a linear integral equation for the self-consistent (effective) field in the medium, consisting of the medium in which many ($M \rightarrow \infty$) small particles are embedded, if suitable physical assumptions are made, which include the following assumptions:

$$a \ll \lambda, \quad a \ll d, \quad (1)$$

MSC: 35J05, 35P25, 73D25, 81U10, 82D20

PACS: 0304K, 43.20.tg, 62.30.td

key words: wave scattering, small particles, many-body problems, metamaterials

where a is the characteristic size of a small particle, λ is the wavelength in the medium, d is the smallest distance between any two distinct particles. We prove that the embedded particles create a new material whose refraction coefficient (in the limit $M \rightarrow \infty$) can be an arbitrary desired function, although the total volume of the embedded particles tends to zero as $M \rightarrow \infty$. Thus, our theory may lead to a new technology in creating materials with desired properties by embedding into original material many small particles with the number of particles per unit volume around any point x as well as their impedances calculated so that the resulting new material would have a desired refraction coefficient $n(x)$. The embedding of the small particles can be done using nanotechnology.

For the theory of wave scattering by small bodies, originated by Rayleigh in 1871, we refer to [2], [1] and [6].

2 Statement of the problem and its solution.

Let $D \subset \mathbb{R}^3$ be a bounded domain filled with a known material with refraction coefficient $n_0(x)$, $x \in \mathbb{R}^3$. The scattering problem consists of finding the solution to the equation

$$\mathcal{L}u := [\nabla^2 + k^2 n_0(x)]u = 0 \text{ in } \mathbb{R}^3, \quad (2)$$

$$u = u_0 + A_0(\beta, \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \beta := \frac{x}{r}, \quad (3)$$

$$u_0 = e^{ik\alpha \cdot x}, \quad (4)$$

$\alpha \in S^2$ is the unit vector in the direction of the incident wave, β is the unit vector in the direction of the scattered wave, $k > 0$ is fixed throughout the paper, $A_0(\beta, \alpha)$ is the scattering amplitude, its dependence on k is not shown since k is fixed.

We assume that

$$n_0(x) = 1 \text{ in } D' := \mathbb{R}^3 \setminus D, \quad (5)$$

and write the Schrödinger equation, equivalent to (2):

$$\mathcal{L}u = [\nabla^2 + k^2 - q_0(x)]u = 0 \text{ in } \mathbb{R}^3, \quad q_0(x) := k^2(1 - n_0(x)) = 0 \text{ in } D'. \quad (6)$$

We assume that q_0 is a bounded piecewise-continuous in \mathbb{R}^3 function. Problem (3), (4), (6) has a unique solution for any square-integrable real-valued $q_0 \in L^2(D)$, or for complex-valued q with $\text{Im } q \leq 0$. We sketch a proof in the Appendix.

Denote by u_0 the solution to the scattering problem (2)-(3), and by G the corresponding Green's function, which solves the following problem:

$$\mathcal{L}G = -\delta(x - y) \text{ in } \mathbb{R}^3, \quad (7)$$

$$\frac{\partial G}{\partial |x|} - ikG = o\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty, \quad (8)$$

where $\delta(x - y)$ in (7) is the delta function. Consider now M small particles D_m , $1 \leq m \leq M$, embedded in D , and the scattering problem:

$$\mathcal{L}\mathcal{U} = 0 \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \quad (9)$$

$$\mathcal{U} = \mathcal{U}_0 + V, \quad \frac{\partial V}{\partial |x|} - ikV = o\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty, \quad (10)$$

$$\frac{\partial \mathcal{U}}{\partial |x|} = h_m \mathcal{U} \text{ on } S_m, \quad 1 \leq m \leq M, \quad (11)$$

where N is the unit exterior normal to S_m , S_m is the boundary of D_m . We assume that S_m is Lipschitz uniformly with respect to m . In the impedance boundary condition (11) the parameter h_m is a constant, possibly complex-valued, such that problem (9)-(11) has a unique solution. For example, this is the case if $\text{Im } h_m \leq 0$.

Let us look for this solution of the form

$$\mathcal{U} = \mathcal{U}_0 + \sum_{m=1}^M \int_{S_m} G(x, t) \sigma_m(t) dt, \quad (12)$$

where σ_m should be chosen so that the boundary condition (11) is satisfied. For arbitrary $\sigma_m \in L^2(S_m)$ the function (12) solves equation (9) and satisfies conditions (10). Therefore, if σ_m are found so that conditions (11) are satisfied, then (12) solves problem (9)-(11). So far we did not use the assumption that D_m are small. Let us assume (1), where

$$a = \frac{1}{2} \max_{1 \leq m \leq M} \text{diam } D_m, \quad d = \min_{1 \leq m, j \leq M, j \neq m} \text{dist}(D_m, D_j), \quad (13)$$

and $\lambda = \frac{2\pi}{k\sqrt{n_0}}$ is the wavelength in the medium with the refraction coefficient $n_0(x)$. We assume that $n_0(x)$ is practically constant on the distances of the order λ .

Let us denote

$$Q_m := \int_{S_m} \sigma_m(t) dt, \quad 1 \leq m \leq M, \quad (14)$$

and rewrite (12) as:

$$\mathcal{U} = \mathcal{U}_0 + \sum_{m=1}^M G(x, x_m) Q_m + \sum_{m=1}^M \int_{S_m} [G(x, t) - G(x, x_m)] \sigma_m(t) dt, \quad (15)$$

where $x_m \in D_m$ can be chosen arbitrary because D_m is small. One may take x_m to be the gravity center of D_m . The gravity center x_m may lie outside D_m if D_m is not convex, but x_m belongs to the convex hull of D_m . We assume that D_m are convex, but this assumption is not essential. If Q_m does not vanish we expect that

$$|G(x, x_m) Q_m| \gg \left| \int_{S_m} [G(x, t) - G(x, x_m)] \sigma_m(t) dt \right|, \quad \text{if } |x - x_m| > d \gg a. \quad (16)$$

We assume that (16) holds for $|x - x_m| \gg a$ if $Q_m \neq 0$ because

$$G(x, t) - G(x, x_m) = \nabla_y G(x, y) \Big|_{y=x_m+\tau(t-x_m)} \cdot (t - x_m), \quad 0 < \tau < 1, \quad (17)$$

so that, for a fixed $k > 0$, one has

$$|G(x, t) - G(x, x_m)| = O\left(\frac{a}{|x - x_m|}\right) \ll 1. \quad (18)$$

Therefore the integral in (16) is $O(ka|G(x, x_m)Q_m|)$ if $Q_m \neq 0$, $|x - x_m| \gg a$, and $ka \ll 1$.

In this approximation we have

$$\mathcal{U}(x) = \mathcal{U}_0(x) + \sum_{m=1}^M G(x, x_m)Q_m, \quad |x - x_m| > d \gg a \quad \forall m, \quad (19)$$

with an error of order $O(ka + \frac{a}{d})$.

Since the potential $q_0(x)$ is known, one may consider the functions $\mathcal{U}_0(x)$ and $G(x, y)$ known. Therefore the scattering problem (9)-(11) is solved if one finds Q_m , $1 \leq m \leq M$.

Let us derive the equations for finding Q_m , $1 \leq m \leq M$, using the boundary condition (11). Denote the effective field, acting on the j -th particle by \mathcal{U}_e , where

$$\mathcal{U}_e(x) := \begin{cases} \mathcal{U}_0(x) + \sum_{m \neq j}^M G(x, x_m)Q_m, & |x - x_j| \sim a, \\ \mathcal{U}_0(x) + \sum_{m=1}^M G(x, x_m)Q_m, & |x - x_m| > d \gg a \quad \forall m. \end{cases} \quad (20)$$

Condition (11) yields:

$$\left(\frac{\partial}{\partial N} - h_j\right) \int_{S_j} G(x, t) \sigma_j(t) dt = - \left(\frac{\partial}{\partial N} - h_j\right) \mathcal{U}_e(x), \quad (21)$$

where x is a point on S_j and the normal derivative on S_j is taken from the exterior to D_j domain in (21) and below. Using formulas (A.8) and (A.15) from the Appendix one may replace the function $G(x, t)$ in the integral in equation (21) by the function $g_0(x, t) = \frac{1}{4\pi|x-t|}$ with the error of order $O(ka)$. Using the known formula for the normal derivative of a single-layer potential (see, e.g., [6, p.5]):

$$\frac{\partial}{\partial N} \int_{S_j} g_0(x, t) \sigma_j(t) dt = \frac{A_j \sigma_j - \sigma_j}{2}, \quad A_j \sigma_j := 2 \int_{S_j} \frac{\partial g_0(s, t)}{\partial N_s} \sigma_j(t) dt, \quad (22)$$

one rewrites (21) as

$$\sigma_j(s) = A_j \sigma_j - 2h_j T_j \sigma_j + 2 \frac{\partial \mathcal{U}_e(s)}{\partial N} - 2h_j \mathcal{U}_e(s), \quad s \in S_j, \quad (23)$$

where

$$T_j \sigma_j := \int_{S_j} g_0(s, t) \sigma_j(t) dt. \quad (24)$$

It is known (see [6, p.96]) that

$$\int_{S_j} ds A_j \sigma = - \int_{S_j} \sigma dt. \quad (25)$$

Integrate (23) over S_j , use (14) and (25), and take into account that

$$\int_{S_j} \frac{\partial \mathcal{U}_e}{\partial N} ds = \int_{D_j} \Delta \mathcal{U}_e dx \approx V_j \Delta \mathcal{U}_e(x_j), \quad V_j = |D_j|, \quad (26)$$

where we have used the smallness of D_j to replace the integral approximately by the expression $V_j \Delta \mathcal{U}_e(x_j)$ and denoted by $|D_j|$ the volume V_j of D_j . Thus, the integration of (23) over S_j yields:

$$Q_j = -h_j \int_{S_j} ds \int_{S_j} \frac{\sigma_j(t) dt}{4\pi|s-t|} - h_j \int_{S_j} \mathcal{U}_e(x) ds + V_j \Delta \mathcal{U}_e(x_j). \quad (27)$$

Note that

$$\int_{S_j} dt \sigma_j(t) \int_{S_j} \frac{ds}{4\pi|s-t|} \approx \int_{S_j} dt \sigma_j(t) \frac{1}{|S_j|} \int_{S_j} dt \int_{S_j} \frac{ds}{4\pi|s-t|} = Q_j \frac{J}{4\pi|S_j|}, \quad (28)$$

where $|S_j|$ is the surface area of S_j ,

$$J := \int_{S_j} \int_{S_j} \frac{ds dt}{|s-t|}, \quad Q_j = \int_{S_j} \sigma_j(t) dt, \quad (29)$$

and we have replaced the function $\int_{S_j} \frac{ds}{4\pi|s-t|}$, practically constant at the distances of order a , by its mean value

$$\frac{1}{|S_j|} \int_{S_j} dt \int_{S_j} \frac{ds}{4\pi|s-t|}$$

i.e., by $\frac{J}{4\pi|S_j|}$. Let us estimate the order of smallness of various terms in (27). The term $\int_{S_j} \mathcal{U}_e(x) ds \approx \mathcal{U}_e(x_j) |S_j| = O(a^2)$, while the term $V_j \Delta \mathcal{U}_e(x_j) = O(a^3)$, if we assume that \mathcal{U}_e and $\Delta \mathcal{U}_e$ are bounded. Therefore, we neglect the last term on the right in (27), and obtain

$$Q_j = - \frac{h_j |S_j| \mathcal{U}_e(x_j)}{1 + \frac{h_j J}{4\pi|S_j|}}. \quad (30)$$

In [6, p.27], the following approximate formula is derived for the electric capacitance D_j of the perfect conductor with the surface S_j :

$$C_j \simeq \frac{4\pi|S_j|^2}{J}, \quad (31)$$

where the conductor is placed in a medium with the dielectric constant $\varepsilon_0 = 1$. Using (31), one may rewrite (30) as:

$$Q_j = -\frac{C_j}{1 + \frac{C_j}{h_j|S_j|}}\mathcal{U}_e(x_j) := -\tilde{C}_j\mathcal{U}_e(x_j), \quad \tilde{C}_j := \frac{C_j}{1 + \frac{C_j}{h_j|S_j|}}. \quad (32)$$

When $h_j \rightarrow \infty$, that is, when the impedance boundary condition becomes the Dirichlet condition in the limit $h_j \rightarrow \infty$, then one obtains from (32) the familiar relation $Q_j = -C_j\mathcal{U}_e(x_j)$ for the total charge Q_j on the surface of the perfect conductor D_j charged to the potential $-\mathcal{U}_e(x_j)$. Here $\mathcal{U}_e(x_j)$ is defined in (20) and on the distances of order a the field $\mathcal{U}_e(x)$ is practically constant.

Substitute (32) into (20), multiply by C_j , set $x = x_j$, and get

$$Q_j = -\tilde{C}_j\mathcal{U}_0(x_j) - \sum_{m \neq j}^M G(x_j, x_m)\tilde{C}_jQ_m, \quad 1 \leq j \leq M. \quad (33)$$

This is a linear algebraic system for finding the unknown Q_m , $1 \leq m \leq M$. The matrix of this system is diagonally dominant if

$$\max_{1 \leq j \leq M} \sum_{m \neq j} |G(x_j, x_m)| |\tilde{C}_j| < 1. \quad (34)$$

If condition (34) holds, then system (33) can be solved by iterations:

$$Q_j^{(n+1)} = -\tilde{C}_j\mathcal{U}_0(x_j) - \sum_{m \neq j}^M G(x_j, x_m)\tilde{C}_j Q_m^{(n)}, \quad Q_j^{(0)} = -\tilde{C}_j\mathcal{U}_0(x_j), \quad (35)$$

and this iterative process converges at the rate of a geometric series. Therefore, system (33) is convenient for solving the scattering problem (9)-(11) when the number of small particles is not very large, $M = O(10^3)$. If this number is very large ($M \sim 10^{23}$), then we study the limiting behavior of $\mathcal{U}_e(x)$ as $M \rightarrow \infty$ and derive an integral equation for the effective field \mathcal{U}_e in the resulting continuous medium.

We rewrite the second line of (20) using formula (32) and obtain the following representation for $\mathcal{U}_e(x)$:

$$\mathcal{U}_e(x) = \mathcal{U}_0(x) - \sum_{m=1}^M G(x, x_m)\tilde{C}_m\mathcal{U}_e(x_m). \quad (36)$$

Assume now that the limiting density of the quantities \tilde{C}_m exists in the following sense: if $\tilde{D} \subset D$ is an arbitrary subdomain of D , then there exists the following limit:

$$\int_{\tilde{D}} \tilde{C}(x)dx = \lim_{M \rightarrow \infty} \sum_{D_m \subset \tilde{D}} \tilde{C}_m. \quad (37)$$

Under this assumption one can pass to the limit $M \rightarrow \infty$ in (36) and get

$$\mathcal{U}_e(x) = \mathcal{U}_0(x) - \int_D G(x, y) \tilde{C}(y) \mathcal{U}_e(y) dy. \quad (38)$$

Applying the operator \mathcal{L} , defined in (6), to (38) and using (7), one gets

$$\mathcal{L}\mathcal{U}_e - \tilde{C}(x)\mathcal{U}_e = 0. \quad (39)$$

This is a Schrödinger equation with the potential

$$q(x) := q_0(x) + \tilde{C}(x). \quad (40)$$

The corresponding scattering amplitude is

$$A(\beta, \alpha) = A_0(\beta, \alpha) - \frac{1}{4\pi} \int_D \mathcal{U}_0(y, -\beta) \tilde{C}(y) \mathcal{U}_e(y) dy, \quad \mathcal{U}_e(y) = \mathcal{U}_e(y, \alpha), \quad (41)$$

where $\alpha \in S^2$ is the unit vector in the direction of the incident plane wave, the following formula (see [5, p.25, formula(5.1.7)]) was used:

$$G(x, y) = \frac{e^{ik|x|}}{4\pi|x|} \mathcal{U}_0(y, -\alpha) + o\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty, \quad \frac{x}{|x|} = \alpha, \quad (42)$$

and $\mathcal{U}_0(x, \alpha)$ is the scattering solution, i.e., the solution to problem (2)-(4) (or, which is the same, to problem (3), (4), (6)).

To calculate the scattering amplitude by formula (41) one has to first solve the integral equation (38) for \mathcal{U}_e . The amplitude $A_0(\beta, \alpha)$ in (41) is the same as in formula (3).

If the number of particles is not very large, one may use formula (20) and calculate the scattering amplitude by the formula

$$A(\beta, \alpha) = A_0(\beta, \alpha) + \frac{1}{4\pi} \sum_{m=1}^M \mathcal{U}_0(x_m, -\beta) Q_m, \quad Q_m = Q_m(\alpha), \quad (43)$$

where the quantities Q_m are found from the linear algebraic system (33).

3 Possible applications to constructing metamaterials.

Our idea for creating a material with a desired refraction coefficient is simple: such a material can be obtained by embedding many small particles in the original material, which fills the domain D . The embedding should create the desired refraction coefficient

$$n(x) := 1 - k^{-2}q(x), \quad \text{where} \quad q(x) = q_0(x) + \tilde{C}(x), \quad (44)$$

compare with formula (6). The function $\tilde{C}(x)$ is fairly general, so the new material, that we have obtained from the original one, which has refraction coefficient $n_0(x)$, is rather general.

Let us prove that the total volume of the embedded small particles is negligible compared with the volume of the original material in the domain D , although the effect, produced by these particles on the refraction coefficient, is large. Consider a unit cube in D , filled with the original material. If the distance between two distinct small particles is not less than d , then the number of these particles in this unit cube is not greater than $\frac{1}{d^3}$, and the total volume of the embedded small particles in this cube is $O(\frac{a^3}{d^3})$. If $M \rightarrow \infty$ and $\frac{a}{d} \rightarrow 0$, then $\frac{a^3}{d^3} \rightarrow 0$, so that the limit of the relative volume of the embedded small particles per unit volume of the original material in D is zero.

How is it possible that these particles produce large effect on the refraction coefficient? How is it possible that $\tilde{C}(y) \neq 0$?

The reason is simple. Let us give a simple calculations in order to explain this reason. Suppose, for simplicity, that the small particles in a unit volume of D are identical. Then the limit (37) exists and is not zero, provided that the limit

$$\lim_{M \rightarrow \infty} \frac{a}{d^3} = \tilde{C}$$

exists and $\tilde{C} \neq 0$. Indeed, $O(\frac{1}{d^3})$ is the order of the number of small particles per unit volume, and, by equation (32),

$$\tilde{C}_m = \frac{C_m}{1 + \frac{C_m}{h_m |S_m|}} = O(a), \quad (45)$$

because $C_m = O(a)$ and the quantity $C_m h_m |S_m|$ can be made $O(1)$ by choosing h_m properly. Note that the dimension $[h_m] = L^{-1}$, where L stands for length, $[C_m] \sim a$, $[a] = L$, $[S_m] \sim a^2$, $[S_m] = L^2$, so the quantity $\frac{C_m}{h_m |S_m|}$ is dimensionless.

To summarize:

One can have $\lim_{d \rightarrow 0} (\frac{a}{d})^3 = 0$ and $\lim_{d \rightarrow 0} \frac{a}{d^3} = \tilde{C} \neq 0$, provided that $d = O(a^{1/3} \gamma)$, where γ is a bounded coefficient whose dimension is $L^{2/3}$, so that the two quantities, d and $a^{1/3} \gamma$, have the same dimension L .

Suppose that all the embedded in D small particles have the same shape and, possibly, different impedances h_m . Let $N(x)$ be the density of the number of these particles per unit volume around point x , that is, for any subdomain $\tilde{D} \subset D$ the following limit exists:

$$\int_{\tilde{D}} N(x) dx = \lim_{M \rightarrow \infty} \sum_{D_m \subset \tilde{D}} 1. \quad (46)$$

One can also write

$$N(x) dx = \sum_{D_m \subset dx} 1,$$

where dx is a small element of volume around point $x \in D$, such that dx still contains many small particles. Then

$$\tilde{C}(x) \approx N(x)\tilde{C}_m = \frac{N(x)C}{1 + \frac{C}{h(x)|S|}}. \quad (47)$$

Here C is the electric capacitance of a perfect conductor with the shape of a single small particle, $|S|$ is the surface area of this particle, and $h(x)$ is the boundary impedance of small particles around point x . If $h(x) = h_1(x) + ih_2(x)$, where h_1 and h_2 are arbitrary real-valued functions, then the function $\tilde{C}(x)$ is also arbitrary. In formula (47) the three function $N(x) \geq 0$, $h_1(x)$, and $h_2(x)$ can be chosen to produce a desired function $\tilde{C}(x)$, that is, to produce the desired refraction coefficient $n(x)$ by formula (44).

Let us give the condition on h_2 and $q_0(x)$ that are sufficient for the uniqueness of the solution to the scattering problem (9)- (11). In Appendix some sufficient conditions for this uniqueness have been established, namely:

$$\text{Im } q(x) \leq 0, \quad \text{Im } h = h_2 \leq 0. \quad (48)$$

If $\tilde{C}(x) = \tilde{C}_1(x) + i\tilde{C}_2(x)$, where $\tilde{C}_1(x)$ and $\tilde{C}_2(x)$ are real-valued functions, then solving (47) for $h(x)$ yields:

$$h(x) = \frac{C\tilde{C}(x)}{|S|[N(x)C - \tilde{C}(x)]}, \quad \text{Im } h = \frac{C}{|S|} \frac{\tilde{C}_2 C N(x)}{[N(x)C - \tilde{C}_1(x)]^2 + \tilde{C}_2^2(x)}. \quad (49)$$

Thus, if $\text{Im } h \leq 0$ then $\text{Im } \tilde{C}_2 \leq 0$. Inequality $\text{Im } q \leq 0$ holds if $\text{Im } q_0(x) + \tilde{C}_2(x) \leq 0$. Therefore, if one wants to create a material with the desired refraction coefficient $n(x)$, i.e., with a desired $q(x) = k^2 - k^2 n(x)$, $\text{Im } q(x) \leq 0$, then one starts with an arbitrary $n_0(x)$, $\text{Im } n_0 = 0$, i.e., with $q_0(x) = k^2 - k^2 n_0(x)$. Given $q_0(x)$ and $q(x)$, one finds $\tilde{C}(x) = q(x) - q_0(x)$, and then uses formula (49) for finding $N(x)$ and $h(x)$ from $\tilde{C}_1(x)$ and $\tilde{C}_2(x)$. The function $N(x) \geq 0$ in (49) is the number of small particles per unit volume around point $x \in D$. We want to prove that one can choose the functions $h_1(x)$ and $h_2(x)$ so that $N(x)$ is positive, $\tilde{C}_2 \leq 0$, \tilde{C}_1, \tilde{C}_2 are given functions, and (47) holds.

Let

$$\frac{|S|}{C} := b > 0, \quad H := bh(x) = H_1 + iH_2. \quad (50)$$

Then the first equation (49) implies

$$CN(x) = \frac{\tilde{C}_1 + i\tilde{C}_2}{H} + \tilde{C}_1 + i\tilde{C}_2. \quad (51)$$

Since $CN(x) > 0$, and $N(x) > 0$, the real part of the right side of (51) should be positive:

$$\frac{\tilde{C}_1 H_1 + \tilde{C}_2 H_2}{H_1^2 + H_2^2} + \tilde{C}_1 > 0, \quad (52)$$

and the imaginary part should vanish:

$$\frac{\tilde{C}_2 H_1 - \tilde{C}_1 H_2}{H_1^2 + H_2^2} + \tilde{C}_2 = 0. \quad (53)$$

Condition (53) holds if

$$\tilde{C}_1 = \tilde{C}_2 \frac{H_1^2 + H_2^2 + H_1}{H_2}, \quad H_2 \neq 0. \quad (54)$$

Using (54), write (52) as

$$\frac{\tilde{C}_2}{H_2} \cdot \left[\frac{(H_1^2 + H_2^2 + H_1)^2 + H_2^2}{H_1^2 + H_2^2} \right] > 0. \quad (55)$$

It follows from (55) that H_2 and \tilde{C}_2 are of the same sign. We may be interested in the materials for which the solution of the scattering problem is unique. Therefore we wish to satisfy conditions (48). The argument below shows that if $\text{Im } q_0 = 0$ and $\text{Im } q < 0$, then $\tilde{C}_2 < 0$.

Indeed, if $\text{Im } q_0 = 0$ and $\text{Im } q < 0$, then $H_2 < 0$, and (55) implies $\tilde{C}_2 < 0$. Conversely, if $\tilde{C}_2 < 0$ and $\text{Im } q_0 = 0$, then (55) implies $H_2 < 0$. We conclude that if $H_2 < 0$, and H_1, H_2 satisfy (54), where $\tilde{C}_2 < 0$ and \tilde{C}_1 are given, then the number $N(x)$ of small particles per unit volume around point $x \in D$, calculated by formula (51), is nonnegative (it can vanish around some points x), so that it has physical meaning.

If one embeds small particles of the same shape with the density of their numbers $N(x)$ and chooses the boundary impedance $h(x)$ of these particles so that the function $bh(x) := H(x) := H_1 + iH_2$ satisfies the condition $H_2(x) < 0$, and if equation (51) holds, then the material one obtains by the embedding of these small particles into D will have the desired refraction coefficient $n(x)$, and the relative total volume of these particles will be negligible.

Appendix

1. *Uniqueness of the solution to the problem (3), (4), (6).*

If u_1 and u_2 solve (3), (4), (6), then $v := u_1 - u_2$ solves (6) and satisfies the radiation condition:

$$\frac{\partial v}{\partial r} = ikv + o\left(\frac{1}{r}\right). \quad (\text{A.1})$$

Multiply (6), where v replaces u , by \bar{v} , the overbar stands for complex conjugate, take complex conjugate of (6) with v replacing u , multiply it by v , subtract from the first equation, use Green's formula, and get

$$\int_{|x|=r} (\bar{v}v_r - v\bar{v}_r)ds - \int_D (q_0 - \bar{q}_0)|v|^2 dx = 0. \quad (\text{A.2})$$

Use (A.1) and let $r \rightarrow \infty$ in (A.2). This yields

$$-2i \operatorname{Im} \int_D q_0(x) |v|^2 dx + 2i k \int_{S^2} |A(\beta)|^2 d\beta = 0, \quad (\text{A.3})$$

where

$$v(\beta r) = A(\beta) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty. \quad (\text{A.4})$$

If $\operatorname{Im} q_0 \leq 0$, then (A.3) implies $A(\beta) = 0$.

This, in turn, implies that v satisfies the following relations:

$$(\nabla^2 + k^2)v = 0 \quad \text{in} \quad D' := \mathbb{R}^3 \setminus D, \quad \lim_{r \rightarrow \infty} \int_{|x|=r} |v|^2 ds = 0. \quad (\text{A.5})$$

From (A.5) one concludes $v = 0$ in D' (see, e.g., Lemma 1 in [3, p.25]). This and the unique continuation property for the solution of the homogeneous elliptic equation (6) imply that $v = 0$ in \mathbb{R}^3 . The proof is complete.

2. *Uniqueness of the solution to problem (9)-(11).*

We argue as above and get

$$-2i \operatorname{Im} \int_D q_0(x) |v|^2 dx + 2ik \int_{S^2} |A(\beta)|^2 d\beta - 2i \sum_{m=1}^M \int_{S_m} \operatorname{Im} h_m |v|^2 ds = 0, \quad k > 0. \quad (\text{A.6})$$

If $\operatorname{Im} h_m \leq 0$, and $\operatorname{Im} q_0(x) \leq 0$, then (A.6) implies $A(\beta) = 0$, $v|_{S_m} = 0$, and this implies $v = 0$ in $\mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m$. This implies $v = 0$ in \mathbb{R}^3 by the argument, given at the end of n.1 of this Appendix.

3. *Estimates of $G(x, y)$ as $|x - y| \rightarrow 0$ and as $|x - y| \rightarrow \infty$.*

We start with the usual integral equation for G :

$$G(x, y) = g(x, y) - \int_D g(x, t) q_0(t) G(t, y) dt, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (\text{A.7})$$

Equation (A.7) implies

$$G(x, y) = g(x, y)[1 + O(|x - y|)], \quad \text{as} \quad |x - y| \rightarrow 0, \quad (\text{A.8})$$

provided that

$$\sup_{x, y \in \mathbb{R}^3} \int_D \frac{|q_0(t)| dt}{|x - t||t - y|} \leq c, \quad (\text{A.9})$$

where $c > 0$ stands for various constants. We assume that (A.9) holds. For example, (A.9) holds if q_0 is bounded. We also have

$$\nabla_y G(x, y) = \nabla_y g(x, y) - \int_D g(x, t) q_0(t) \nabla_y G(t, y) dt. \quad (\text{A.10})$$

This implies

$$\nabla_y G(x, y) = \nabla_y g(x, y) \left[1 + O \left(|x - y|^2 \ln \frac{1}{|x - y|} \right) \right] \quad \text{as } |x - y| \rightarrow 0. \quad (\text{A.11})$$

Indeed,

$$\nabla_y g(x, y) = ik g(x, y) \left(1 - \frac{1}{ik|x - y|} \right) \frac{y - x}{|y - x|}, \quad (\text{A.12})$$

so

$$|\nabla_y g(x, y)| = O(|x - y|^{-2}), \quad \text{as } |x - y| \rightarrow 0, \quad (\text{A.13})$$

and

$$\left| \int_D g(x, t) q_0(t) \nabla_y G(t, y) dt \right| \leq c \int_D \frac{dt}{|x - t||t - y|^2} \leq c \ln \frac{1}{|x - y|} \quad \text{as } |x - y| \rightarrow 0. \quad (\text{A.14})$$

Note that

$$g(x, y) = g_0(x, y)[1 + O(ka)], \quad g_0(x, y) := \frac{1}{4\pi|x - y|}, \quad \text{if } |x - y| < a. \quad (\text{A.15})$$

Now, let $|x - y| \rightarrow \infty$. Assume that y is in a bounded domain. Then (A.7) implies that

$$|G(x, y)| + |\nabla_y G(x, y)| = O \left(\frac{1}{|x|} \right), \quad \text{if } |x| \rightarrow \infty, \quad |y| \leq c. \quad (\text{A.16})$$

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